

# FREDHOLM CRITERIA FOR PSEUDODIFFERENTIAL OPERATORS AND INDUCED REPRESENTATIONS OF GROUPOID ALGEBRAS

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ABSTRACT. We characterize the groupoids for which an operator is Fredholm if, and only if, its principal symbol and all its boundary restrictions are invertible. A groupoid with this property is called *Fredholm*. Using results on the Effros-Hahn conjecture, we show that an almost amenable, Hausdorff, second countable groupoid is Fredholm. Many groupoids, and hence many pseudodifferential operators appearing in practice, fit into this framework. We show that the desingularization of groupoids preserves the class of Fredholm groupoids.

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## 1. INTRODUCTION

We obtain necessary and sufficient conditions for operators modeled by groupoids to be Fredholm. Examples include operators obtained by desingularization of singular spaces by successively blowing up the lowest dimensional singular strata. We begin with a general study of Fredholm conditions for pseudodifferential operators in the framework of Fredholm groupoids. A *Fredholm groupoid* is, by definition, a locally compact groupoid with a Haar system for which the Fredholm property is equivalent to the invertibility of the principal symbol and of its fiberwise boundary restrictions. We obtain a general characterization of Fredholm groupoids. In particular, using some results of Renault [31, 32] and Ionescu and Williams [13], we show that an almost amenable, second-countable, Hausdorff groupoid is Fredholm.

Let  $\mathcal{G}$  be a groupoid with base  $M$  modeling the analysis on some singular space. An  $A(\mathcal{G})$ -*tame submanifold*  $L \subset M$  is one that has, by definition, a tubular neighborhood on which  $A(\mathcal{G})$  becomes a pull-back Lie algebroid. The “desingularization”  $[[\mathcal{G} : L]]$  of  $\mathcal{G}$  along  $L$  [24] is the a groupoid modeling the analysis on the space obtained by blowing-up  $L$ . The space of units of the desingularization  $[[\mathcal{G} : L]]$  is  $[M : L]$ , the blow-up of  $M$  along  $L$ . The desingularization groupoid is not a blown-up space, however. We use the explicit structure of the desingularized groupoid  $[[\mathcal{G} : L]]$  (see [24]) to show that it is Fredholm if  $\mathcal{G}$  is. Our results specialize to yield Fredholm conditions for operators on manifolds with cylindrical and poly-cylindrical ends, on manifolds that are asymptotically Euclidean or asymptotically hyperbolic, on products of such manifolds, on manifolds that locally at infinity are products of such manifolds, and on others. Most of the (generally) easy proofs are contained in [25], this paper being a summary and update of some results in that paper. We thank Ingrid and Daniel Belitiță, Claire Debord, Siegfried

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## 2. FREDHOLM GROUPOIDS

Recall that a *groupoid*  $\mathcal{G}$  is a small category in which every morphism is invertible. We shall write  $\mathcal{G} \rightrightarrows M$  for a groupoid with objects (or *units*)  $M$ . The domain and range of a morphism therefore give rise to maps  $d, r : \mathcal{G} \rightarrow M$ . We refer to [21, 30] for the results and concepts used—but not recalled—in this paper. Let  $\mathcal{G} \rightrightarrows M$  be a locally compact groupoid endowed with a Haar system  $(\lambda_x)_{x \in M}$ . We denote by  $C^*(\mathcal{G})$  the  $C^*$ -algebra of  $\mathcal{G}$  and by  $C_r^*(\mathcal{G})$  the *reduced*  $C^*$ -algebra of  $\mathcal{G}$ . Also, we denote  $\mathcal{G}_A := d^{-1}(A)$  and  $\mathcal{G}_A^B := d^{-1}(A) \cap r^{-1}(B)$ . If  $A$  is invariant, in the sense that  $\mathcal{G}_A = \mathcal{G}^A := r^{-1}(A)$ , then  $\mathcal{G}_A$  is a groupoid. As usual, we associate to any  $x \in M$  the *regular* representation  $\pi_x : C^*(\mathcal{G}) \rightarrow \mathcal{B}(L^2(\mathcal{G}_x, \lambda_x))$

$$\pi_x(\varphi)\psi(g) := \varphi * \psi(g) := \int_{\mathcal{G}_{d(g)}} \varphi(gh^{-1})\psi(h)d\lambda_{d(g)}(h), \quad \phi \in \mathcal{C}_c(\mathcal{G}).$$

All the morphisms (and representations) in this paper will preserve the involution.

*Example 2.1.* Recall that the *pair groupoid*  $\mathcal{H} := A \times A$  is the groupoid having *exactly* one arrow between any two units. Let  $\mathcal{G} \rightrightarrows L$  be a groupoid and  $f : M \rightarrow L$ . An important generalization of the pair groupoid is the *fibered pull-back groupoid*:

$$f^{\downarrow\downarrow}(\mathcal{G}) := \{(m, g, m') \in M \times \mathcal{G} \times M, f(m) = r(g), d(g) = f(m')\},$$

with units  $M$  and product  $(m, g, m')(m', g', m'') = (m, gg', m'')$ .

**Definition 2.2.** A locally compact groupoid  $\mathcal{G} \rightrightarrows M$  is called *Fredholm* if:

- (i)  $\mathcal{G}_{M_0} \simeq M_0 \times M_0$  for some open, dense,  $\mathcal{G}$ -invariant subset  $M_0 \subset M$ .
- (ii) For any  $a \in C_r^*(\mathcal{G})$ , we have that  $1 + \pi_{x_0}(a)$ ,  $x_0 \in M_0$ , is Fredholm if, and only if,  $1 + \pi_x(a)$  is invertible for any  $x \in F := M \setminus M_0$ .

Both  $M_0$  and  $F$  are uniquely determined by  $\mathcal{G}$ , so this notation will remain fixed in what follows. Also, in Definition 2.2, all representations  $\pi_{x_0}$ ,  $x_0 \in M_0$ , are unitarily equivalent to the *vector representation*  $\pi_0 : C^*(\mathcal{G}) \rightarrow \mathcal{L}(L^2(M_0))$  obtained by identifying  $r : \mathcal{G}_{x_0} \simeq M_0$ .

Recall [33] that if  $A$  is a  $C^*$ -algebra with unit, then a set  $\mathcal{F}$  of representations of  $A$  is called *invertibility sufficient* if the following condition is satisfied: “ $a \in A$  is invertible if, and only if,  $\phi(a)$  is invertible for all  $\phi \in \mathcal{F}$ .” If  $A$  does not have a unit, we replace  $A$  with  $A^+ := A \oplus \mathbb{C}$  and  $\mathcal{F}$  with  $\mathcal{F}^+ := \mathcal{F} \cup \{\chi_0 : A^+ \rightarrow \mathbb{C}\}$  [26]. The following two results give a first characterization of Fredholm groupoids.

**Theorem 2.3.** *Let  $\mathcal{G} \rightrightarrows M$  be a Fredholm groupoid. Then*

- (i) *The vector representation  $\pi_0 : C_r^*(\mathcal{G}) \rightarrow \mathcal{L}(L^2(M_0))$  is injective.*
- (ii) *The canonical projection induces an isomorphism  $C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_{M_0}) \simeq C_r^*(\mathcal{G}_F)$ .*
- (iii)  *$\{\pi_x, x \in F\}$  is an invertibility sufficient set of representations of  $C_r^*(\mathcal{G}_F)$ .*

For index theory, the first two conditions of the theorem are enough. Invertibility sufficient families of representations consist of non-degenerate representations. A non-degenerate representation of a (closed, two-sided) ideal in a  $C^*$ -algebra has a unique extension to the whole algebra.

The following strong converse of Theorem 2.3 holds true.

**Theorem 2.4.** *Let  $\mathcal{G} \rightrightarrows M$  be a locally compact groupoid satisfying the three conditions (i-iii) of Theorem 2.3. Then, for any unital  $C^*$ -algebra  $\Psi$  containing  $C_r^*(\mathcal{G})$  as an essential ideal and for any  $a \in \Psi$ , we have that  $\pi_0(a)$  is Fredholm if, and only if,  $\pi_x(a)$  is invertible for each  $x \notin M_0$  **and** the image of  $a$  in  $\Psi/C_r^*(\mathcal{G})$  is invertible.*

## 3. RELATION TO THE EFFROS-HAHN CONJECTURE

We now want to obtain some more concrete and easier to use conditions for a groupoid  $\mathcal{G}$  to be Fredholm. We shall say that a locally compact groupoid  $\mathcal{G}$  *has the weak inclusion property* (wi-property, for short) if every irreducible representation of  $C^*(\mathcal{G})$  is *weakly contained* in a representation of  $C^*(\mathcal{G})$  induced from a representation of an isotropy subgroup  $\mathcal{G}_y^y := d^{-1}(y) \cap r^{-1}(y)$  [13, 32] (equivalently, if every primitive ideal of  $C^*(\mathcal{G})$  *contains* an ideal induced from a representation of an isotropy subgroup  $\mathcal{G}_y^y$ ). A groupoid  $\mathcal{G}$  with the wi-property and such that all the groups  $\mathcal{G}_y^y$  are amenable will be called *EH-amenable*. Recall that a locally compact groupoid  $\mathcal{G}$  *satisfies the generalized Effros-Hahn (EH) conjecture* if every primitive ideal of  $C^*(\mathcal{G})$  is *induced* from a representation of an isotropy group  $\mathcal{G}_y^y$  [8, 13, 14, 30, 40].

*Example 3.1.* Let  $\mathcal{G} \rightrightarrows B$  be a locally trivial bundle of groups (so  $d = r$ ) with typical fiber a locally compact group  $G$ . Also, let  $f : M \rightarrow B$  be a continuous map that is a local fibration. Then  $f^{\downarrow}(\mathcal{G})$  is a locally compact groupoid with a Haar system that satisfies the generalized EH conjecture, and hence it has the wi-property. It will be EH-amenable if, and only if, the group  $G$  is amenable.

Recall that a groupoid is called *metrically amenable* if the canonical surjection  $C^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})$  is injective [38]. We shall need two results from [26] (see also [9]).

**Proposition 3.2.** *Let  $\mathcal{G} \rightrightarrows F$  be an EH-amenable locally compact groupoid. Then the family of regular representations  $\{\pi_y, y \in F\}$  of  $C^*(\mathcal{G})$  is invertibility sufficient. In particular,  $\mathcal{G}$  is metrically amenable.*

The class of EH-amenable groupoids is closed under extensions.

**Proposition 3.3.** *Let  $\mathcal{G} \rightrightarrows M$  be a locally compact groupoid,  $M_0 \subset M$  be a  $\mathcal{G}$ -invariant open subset, and  $F := M \setminus M_0$ . Then  $\mathcal{G}$  is EH-amenable if, and only if, both  $\mathcal{G}_F$  and  $\mathcal{G}_{M_0}$  are EH-amenable. The same holds if one replaces “is EH-amenable” with “satisfies the generalized EH conjecture” or “has the wi-property.”*

The following result leads to more applicable Fredholm conditions.

**Proposition 3.4.** *Let  $\mathcal{G} \rightrightarrows M$  be a Hausdorff, locally compact groupoid with an open, dense,  $\mathcal{G}$ -invariant subset  $M_0 \subset M$  such that  $\mathcal{G}_{M_0} \simeq M_0 \times M_0$ . If  $\mathcal{G}$  is EH-amenable, then  $\mathcal{G}$  is Fredholm.*

Let  $U_i \subset U_{i+1} \subset M$  be open,  $\mathcal{G}$ -invariant subsets of  $M$ , with  $U_{-1} = \emptyset$  and  $U_N = M$ . If  $\mathcal{G}_{U_{i+1} \setminus U_i}$  is topologically amenable for all  $i$ , then we shall say that the locally compact groupoid  $\mathcal{G} \rightrightarrows M$  is *almost amenable*. By combining the above two propositions with the proof of the generalized EH conjecture [13, 31, 32] for amenable, Hausdorff, second countable groupoids, we obtain the following result.

**Theorem 3.5.** *Let  $\mathcal{G} \rightrightarrows M$  be an almost amenable, Hausdorff, second countable groupoid. Then  $\mathcal{G}$  satisfies the generalized EH conjecture. If also  $\mathcal{G}_{M_0} \simeq M_0 \times M_0$  for an open, dense,  $\mathcal{G}$ -invariant subset  $M_0 \subset M$ , then  $\mathcal{G}$  is Fredholm.*

We are interested in Fredholm groupoids because of their applications to Fredholm conditions. Let  $\mathcal{G}$  be a continuous family groupoid [15] and  $\Psi^m(\mathcal{G})$  be the space of order  $m$ , classical pseudodifferential operators  $P = (P_x)_{x \in M}$  on  $\mathcal{G}$  [15] (see [2, 1, 22, 27, 39] for Lie groupoids, which are continuous family groupoids). Recall that, by definition, each  $P_x \in \Psi^m(\mathcal{G}_x)$ ,  $x \in M$ . Also,  $P$  acts on  $M_0$  via  $P_{x_0} : H^s(M_0) \rightarrow H^{s-m}(M_0)$ ,  $x_0 \in M_0 = r(\mathcal{G}_{x_0}) \simeq \mathcal{G}_{x_0}$ . The following result is interesting only in the case  $M$  compact.

**Theorem 3.6.** *Let  $\mathcal{G} \rightrightarrows M$  be a Fredholm, continuous family groupoid and let  $M_0 \subset M$  be the dense,  $\mathcal{G}$ -invariant subset such that  $\mathcal{G}_{M_0} \simeq M_0 \times M_0$ . We have*

$$P : H^s(M_0) \rightarrow H^{s-m}(M_0), P \in \Psi^m(\mathcal{G}), \text{ is Fredholm} \Leftrightarrow P \text{ is elliptic and}$$

$$P_x : H^s(\mathcal{G}_x) \rightarrow H^{s-m}(\mathcal{G}_x) \text{ is invertible for all } x \in F := M \setminus M_0.$$

This theorem is proved by considering  $a := (1 + \Delta)^{(s-m)/2} P (1 + \Delta)^{-s/2}$ , which belongs to the closure of  $\Psi^0(\mathcal{G})$ , by the results in [17]. For the next theorem, however, one has to consider the Cayley transform of  $P$  instead of the operator  $a$ .

**Theorem 3.7.** *Let  $\mathcal{G} \rightrightarrows M$  be as in Theorem 3.6 and let  $P \in \Psi^m(\mathcal{G})$  be an elliptic operator. Then its essential spectrum is*

$$\sigma_{ess}(P) = \begin{cases} \cup_{x \in F} \sigma(P_x) & \text{if } m > 0 \\ \cup_{x \in F} \sigma(P_x) \cup \text{Im}(\sigma_0(P)) & \text{if } m \leq 0. \end{cases}$$

The above two theorems extend to operators acting on vector bundles on  $M$ . The operators  $P_x$  are the analogues in our setting of the “limit operators” considered in [5, 28] and many other references. See also [3, 6, 10, 11, 15, 16, 17, 18, 19, 20, 23, 29, 34, 35, 36, 37] and the references therein for related results.

#### 4. DESINGULARIZATION AND FREDHOLM CONDITIONS

We want an ample supply of Fredholm groupoids. In this section, we recall the desingularization procedure along a “tame” submanifold of the set of units of a Lie groupoid [24]. Recall the thick pull-back  $\pi^{\downarrow\downarrow}(B)$  of Example 2.1.

**Definition 4.1.** Let  $A \rightarrow M$  be a Lie algebroid. An  $A$ -tame submanifold  $L \subset M$  is a submanifold that has a tubular neighborhood  $L \subset U$ , such that there exists a Lie algebroid  $B \rightarrow L$  and an isomorphism  $A|_U \simeq \pi^{\downarrow\downarrow}(B)$  over  $U$ ,  $\pi : U \rightarrow L$ .

By  $A(\mathcal{G})$  we denote the Lie algebroid of a Lie groupoid  $\mathcal{G}$ . We have the following structure result in a neighborhood of a tame submanifold.

**Theorem 4.2.** *Let  $\mathcal{G} \rightrightarrows M$  be a Lie groupoid and let  $L \subset M$  be an  $A(\mathcal{G})$ -tame submanifold of  $M$  with tubular neighborhood  $\pi : U \rightarrow L \subset U$ . Assume that the fibers of  $\pi : U \rightarrow L$  are simply-connected. Then the reduction groupoids  $\mathcal{G}_L^L$  and  $\mathcal{G}_U^U$  are Lie groupoids and there exists a natural isomorphism over  $U$  of Lie groupoids*

$$\mathcal{G}_U^U \simeq \pi^{\downarrow\downarrow}(\mathcal{G}_L^L) := U \times_{\pi} \mathcal{G}_L^L \times_{\pi} U.$$

From now on, we shall assume that  $\mathcal{G} \rightrightarrows M$  and  $L \subset M$  are as in Theorem 4.2 and use this theorem to define the desingularization  $[[\mathcal{G} : L]]$ . To construct  $[[\mathcal{G} : L]]$ , we proceed in four steps [24], the first three of which deal with the particular case  $\mathcal{G} = f^{\downarrow\downarrow}(\mathcal{H})$ , where  $f : M = U \rightarrow L$  is a vector bundle. We denote by  $S$  the set of unit vectors in  $U$  for some fixed metric.

**Construction of the desingularization. Step one.** We first consider the adiabatic groupoid  $\mathcal{H}_{ad}$  of  $\mathcal{H}$  [4, 7, 27]. It is a Lie groupoid with units  $L \times [0, \infty)$  and Lie algebroid  $A(\mathcal{H}_{ad}) = A(\mathcal{H}) \times [0, \infty) \rightarrow L \times [0, \infty)$ , which, as a vector bundle, is the pull-back of  $A(\mathcal{H}) \rightarrow L$  to  $L \times [0, \infty) \rightarrow L$ . The Lie algebroid structure on the sections of  $A(\mathcal{H}_{ad})$  is not that of a pull-back, but is given by  $[X, Y](t) = t[X(t), Y(t)]$ . As a set,  $\mathcal{H}_{ad}$  is the disjoint union

$$\mathcal{H}_{ad} := A(\mathcal{H}) \times \{0\} \sqcup \mathcal{H} \times (0, \infty).$$

The groupoid structure of  $\mathcal{H}_{ad}$  is such that  $A(\mathcal{H}) \times \{0\}$  has the Lie groupoid structure of a bundle of Lie groups and  $\mathcal{H} \times (0, \infty)$  has the product Lie groupoid structure (that is  $(0, \infty)$  has only units, and all orbits are reduced to a single point).

*Step two.* Let  $\pi : S \rightarrow L$  be the projection. We denote also by  $\pi$  the resulting map  $S \times [0, \infty) \rightarrow L \times [0, \infty)$ . Then we consider the Lie groupoid  $\pi^{\downarrow\downarrow}(\mathcal{H}_{ad})$ .

*Step three.* Let  $\mathbb{R}_+^* = (0, \infty)$  act by dilations on the  $[0, \infty)$  variable on  $\pi^{\downarrow\downarrow}(\mathcal{H}_{ad})$  and consider the semi-direct product  $\pi^{\downarrow\downarrow}(\mathcal{H}_{ad}) \rtimes \mathbb{R}_+^*$  [7]. We then define

$$(1) \quad [[\mathcal{G} : L]] := \pi^{\downarrow\downarrow}(\mathcal{H}_{ad}) \rtimes \mathbb{R}_+^*, \quad \text{if } \mathcal{G} = \pi^{\downarrow\downarrow}(\mathcal{H}) \text{ for a vector bundle } \pi : M \rightarrow L.$$

*Step three.* Let  $\mathcal{G} \rightrightarrows M$  be a Lie algebroid and  $L \subset M$  be an  $A(\mathcal{G})$ -tame submanifold. Let  $W := M \setminus L$  and let  $U$  be as in Theorem 4.2. Then we glue  $\mathcal{G}_W^W$  and  $[[\mathcal{G}_U^U : L]]$  along the common open subset [12, 24] to obtain  $[[\mathcal{G} : L]]$ .

If  $\mathcal{G}$  is Hausdorff, then  $[[\mathcal{G}_U^U : L]]$  is also Hausdorff, [24]. We denote by  $[M : L]$  the *blow-up* of  $M$  with respect to  $L$ , it is obtained by replacing  $L$  with the set of unit vectors  $S$  of its normal bundle in  $M$ . So  $S = [M : L] \setminus (M \setminus L) = [M : L] \setminus W$ . Let  $\mathcal{G}$  be as in Theorem 4.2. We then have the following structural result for the desingularization  $\mathcal{K} := [[\mathcal{G} : L]]$ .

**Proposition 4.3.** *The desingularization  $\mathcal{K} := [[\mathcal{G} : L]]$  is a Lie groupoid with units  $[M : L]$ . The subset  $S \subset [M : L]$  is closed and  $\mathcal{K}$ -invariant. The restriction  $\mathcal{K}_S = [[\mathcal{G} : L]] \setminus \mathcal{G}_W^W$  is isomorphic to the fibered pull-back  $\pi^{\downarrow}(A(\mathcal{K}) \rtimes \mathbb{R}_+^*)$  to  $S$  via the natural projection  $\pi : S \rightarrow L$ , where  $A(\mathcal{K}) \rtimes \mathbb{R}_+^*$  is regarded as a bundle of Lie groups. The inclusion  $\mathcal{G}_W^W \rightarrow \mathcal{G}$  induces an isomorphism  $C^*(\mathcal{G}_W^W) \simeq C^*(\mathcal{G})$ .*

One sees that the resulting glued set is a Hausdorff groupoid as in [12].

**Theorem 4.4.** *Let  $\mathcal{G} \rightrightarrows M$  be a Lie groupoid. Let us assume that  $\mathcal{G}$  is Fredholm. Let  $L \subset M$  be an  $A(\mathcal{G})$ -tame submanifold. Then  $[[\mathcal{G} : L]]$  is also Fredholm.*

Desingularization preserves the class of Fredholm groupoids.

**Theorem 4.5.** *Let us assume that  $\mathcal{G}$  is obtained from a pair groupoid  $M \times M$  (with  $M$  smooth) by a sequence of desingularizations with respect to tame submanifolds. Then  $\mathcal{G}$  is a Hausdorff Fredholm Lie groupoid.*

This theorem can be used to obtain Fredholm conditions for operators on polyhedral domains, as well as on some other stratified spaces.

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